

# NAG C Library Chapter Introduction

## g05 – Random Number Generators

### Contents

<b>1</b>	<b>Scope of the Chapter</b> .....	2
<b>2</b>	<b>Background to the Problems</b> .....	2
2.1	Pseudo-random Numbers .....	2
2.2	Quasi-random Numbers .....	2
2.3	Non-uniform Random Numbers .....	3
2.4	Copulas and Other Random Structures .....	3
<b>3</b>	<b>Recommendations on Choice and Use of Available Functions</b> .....	4
3.1	Design of the Chapter .....	4
3.1.1	Initialization .....	4
3.1.2	Repeated initialization .....	4
3.1.3	Copulas .....	4
3.1.4	Quasi-random generators .....	4
3.2	Programming Advice .....	4
<b>4</b>	<b>Index</b> .....	5
<b>5</b>	<b>Functions Withdrawn or Scheduled for Withdrawal</b> .....	6
<b>6</b>	<b>References</b> .....	6

## 1 Scope of the Chapter

This chapter is concerned with the generation of sequences of independent pseudo-random and quasi-random numbers from various distributions, and the generation of pseudo-random time series from specified time series models.

## 2 Background to the Problems

### 2.1 Pseudo-random Numbers

A sequence of pseudo-random numbers is a sequence of numbers generated in some systematic way such that its statistical properties are as close as possible to those of true random numbers: for example, negligible correlation between consecutive numbers. The most common methods are based on the **multiplicative congruential** algorithm, see Knuth (1981). The basic algorithm is defined as:

$$n_i = (a \times n_{i-1}) \bmod m \quad (1)$$

The integers  $n_i$  are then divided by  $m$  to give uniformly distributed pseudo-random numbers lying in the interval  $(0, 1)$ .

Alternatively there is a variant known as the Wichmann–Hill algorithm, see Maclaren (1989), defined as:

$$\begin{aligned} n_{1,i} &= (a_1 \times n_{1,i-1}) \bmod m_1 \\ n_{2,i} &= (a_2 \times n_{2,i-1}) \bmod m_2 \\ n_{3,i} &= (a_3 \times n_{3,i-1}) \bmod m_3 \\ n_{4,i} &= (a_4 \times n_{4,i-1}) \bmod m_4 \\ U_i &= \left( \frac{n_{1,i}}{m_1} + \frac{n_{2,i}}{m_2} + \frac{n_{3,i}}{m_3} + \frac{n_{4,i}}{m_4} \right) \bmod 1.0 \end{aligned} \quad (2)$$

This generates pseudo-random numbers  $U_i$ , uniformly distributed in the interval  $(0, 1)$ .

Either of these algorithms can be selected to generate uniformly distributed pseudo-random numbers. If the basic algorithm (1) is selected then the NAG generator uses the values  $a = 13^{13}$  and  $m = 2^{59}$  in (1). This generator gives a **cycle length** (i.e., the number of random numbers before the sequence starts repeating itself) of  $2^{57}$ . A good rule of thumb is never to use more numbers than the square root of the cycle length in any one experiment as the statistical properties are impaired. For closely related reasons, breaking numbers down into their bit patterns and using individual bits may cause trouble.

If the Wichmann–Hill algorithm is selected then one or more of 273 independent generators are available. Each of these is defined by the set of constants  $a_j$  and  $m_j$  for  $j = 1, \dots, 4$ . The constants  $a_j$  are in the range 112 to 127 and the constants  $m_j$  are prime numbers in the range 16718909 to 16776971, which are close to  $2^{24} = 16777216$ . These constants have been chosen so that they give good results with the spectral test, see Knuth (1981) and Maclaren (1989). The period of each Wichmann–Hill generator would be at least  $2^{92}$  if it were not for common factors between  $(m_1 - 1)$ ,  $(m_2 - 1)$ ,  $(m_3 - 1)$  and  $(m_4 - 1)$ . However, each generator should still have a period of at least  $2^{80}$ . Further discussion of the properties of these generators is given in Maclaren (1989) where it was shown that the generated pseudo-random sequences are essentially independent of one another according to the spectral test.

The sequence given in (1) needs an initial value  $n_0$ , known as the **seed**, while the sequence given in (2) needs four such seeds. The use of the same seed will lead to the same sequence of numbers when these are computed serially. One method of obtaining a seed is to use the real-time clock; this will give a non-repeatable sequence. It is important to note that the statistical properties of the random numbers are only guaranteed within sequences and not between sequences. Repeated initialization will thus render the numbers obtained less rather than more independent. Similarly the statistical properties of the random numbers are not guaranteed between two sequences generated using the two algorithms.

### 2.2 Quasi-random Numbers

Quasi-random numbers are intended primarily for use in Monte Carlo integration. Like pseudo-random numbers they are uniformly distributed but they are not statistically independent, rather they are designed to give a more even distribution in multidimensional space (uniformity). Therefore, they are often more efficient than pseudo-random numbers in multidimensional Monte Carlo methods. There are several quasi-

random generators, three of which are available in this chapter, they are the Sobol, Faure and Neiderreiter generators.

### 2.3 Non-uniform Random Numbers

Random numbers from other distributions may be obtained from the uniform random numbers by the use of transformations and rejection techniques, and for discrete distributions, by table based methods.

#### (a) Transformation Methods

For a continuous random variable, if the cumulative distribution function (CDF) is  $F(x)$  then for a uniform  $(0, 1)$  random variate  $u$ ,  $y = F^{-1}(u)$  will have CDF  $F(x)$ . This method is only efficient in a few simple cases such as the exponential distribution with mean  $\mu$ , in which case  $F^{-1}(u) = -\mu \log u$ . Other transformations are based on the joint distribution of several random variables. In the bivariate case, if  $v$  and  $w$  are random variates there may be a function  $g$  such that  $y = g(v, w)$  has the required distribution; for example, the Student's  $t$ -distribution with  $n$  degrees of freedom in which  $v$  has a Normal distribution,  $w$  has a gamma distribution and  $g(v, w) = v\sqrt{n/w}$ .

#### (b) Rejection Methods

Rejection techniques are based on the ability to easily generate random numbers from a distribution (called the envelope) similar to the distribution required. The value from the envelope distribution is then accepted as a random number from the required distribution with a certain probability; otherwise, it is rejected and a new number is generated from the envelope distribution.

#### (c) Table Search Methods

For discrete distributions, if the cumulative probabilities,  $P_i = \text{Prob}(x \leq i)$ , are stored in a table then, given  $u$  from a uniform  $(0, 1)$  distribution, the table is searched for  $i$  such that  $P_{i-1} < u \leq P_i$ . The returned value  $i$  will have the required distribution. The table searching can be made faster by means of an index, see Ripley (1987). The effort required to set up the table and its index may be considerable, but the methods are very efficient when many values are needed from the same distribution.

### 2.4 Copulas and Other Random Structures

A copula is a function that links the univariate marginal distributions with their multivariate distribution. Sklar's theorem Sklar (1973) states that if  $f$  is an  $m$ -dimensional distribution function with continuous margins  $f_1, f_2, \dots, f_m$ , then  $f$  has a unique copula representation,  $c$ , such that

$$f(x_1, x_2, \dots, x_m) = c(f_1(x_1), f_2(x_2), \dots, f_m(x_m))$$

The copula,  $c$ , is a multivariate uniform distribution whose dependence structure is defined by the dependence structure of the multivariate distribution  $f$ , with

$$c(u_1, u_2, \dots, u_m) = f(f_1^{-1}(u_1), f_2^{-1}(u_2), \dots, f_m^{-1}(u_m))$$

where  $u_i \in [0, 1]$ . This relationship can be used to simulate variates from distributions defined by the dependence structure of one distribution and each of the marginal distributions given by another. For additional information see Nelsen (1998) or Boye (Unpublished manuscript) and the references therein.

In addition to random numbers from various distributions, random compound structures can be generated. These include random time series, random matrices and random samples.

The efficiency of a simulation exercise may often be increased by the use of variance reduction methods (see Morgan (1984)). It is also worth considering whether a simulation is the best approach to solving the problem. For example, low-dimensional integrals are usually more efficiently calculated by functions in Chapter d01 rather than by Monte Carlo integration.

## 3 Recommendations on Choice and Use of Available Functions

### 3.1 Design of the Chapter

All the generation functions call an internal generator (selected to be either the basic generator (1) or a Wichmann–Hill generator (2)), which generates random numbers from a uniform distribution over (0, 1).

#### 3.1.1 Initialization

Prior to generating any variates the internal generator must be initialized. Two utility functions are provided for this, `nag_rngs_init_repeatable (g05kbc)` and `nag_rngs_init_nonrepeatable (g05kcc)`, both of which allow either the basic generator or one of the 273 Wichmann–Hill generators to be chosen as the internal generator.

`nag_rngs_init_repeatable (g05kbc)` selects and initializes a generator to a repeatable (when executed serially) state: two calls of `nag_rngs_init_repeatable (g05kbc)` with the same argument-values will result in the same subsequent sequences of random numbers (when both are generated serially). The basic generator or one of 273 Wichmann–Hill generators can be selected as the base generator.

`nag_rngs_init_nonrepeatable (g05kcc)` selects and initializes a generator to a non-repeatable state, in such a way that different calls of `nag_rngs_init_nonrepeatable (g05kcc)`, either in the same run or different runs of the program, will almost certainly result in different subsequent sequences of random numbers.

Functions to save and restore generator states are not required since the **iseed** argument contains the current state and can have its values copied and stored at any time.

#### 3.1.2 Repeated initialization

As mentioned in Section 2, it is important to note that the statistical properties of pseudo-random numbers are only guaranteed within sequences and not between sequences produced by the same generator. Repeated initialization will thus render the numbers obtained less rather than more independent. In a simple case there should be only one call to `nag_rngs_init_repeatable (g05kbc)` or `nag_rngs_init_nonrepeatable (g05kcc)` and this call should be before any call to an actual generation function.

#### 3.1.3 Copulas

After calling `nag_rngs_copula_normal (g05rac)` or `nag_rngs_copula_students_t (g05rbc)` the `g01f` functions in Chapter `g01` can be used to convert the uniform marginal distributors into a different form as required.

#### 3.1.4 Quasi-random generators

The functions available for quasi-random number each provide you with the option of selecting a Sobol, Faure or Neiderreiter sequence. The maximum number of dimensions is 40. If higher dimensions are required these generators can be combined with the pseudo-random generators described above.

`nag_quasi_random_uniform (g05yac)` Uniform distribution

`nag_quasi_random_normal (g05ybc)` Normal distribution

### 3.2 Programming Advice

Take care when programming calls to those functions in this chapter which are functions. The reason is that different calls with the same arguments are intended to give different results.

For example, if you wish to assign to `z` the difference between two successive random numbers generated by `nag_rngs_basic (g05kac)`, beware of writing

```
z = g05kac(igen,iseed) - g05kac(igen,iseed)
```

It is quite legitimate for a C compiler to compile zero, one or two calls to `nag_rngs_basic` (g05kac); if two calls, they may be in either order (if zero or one calls are compiled, `z` would be set to zero). A safe method to program this would be

```
x = g05kac(igen,iseed);
y = g05kac(igen,iseed);
z = x-y
```

## 4 Index

Generating samples, matrices and tables,

random correlation matrix .....	<code>nag_rngs_corr_matrix</code>	(g05qbc)
random orthogonal matrix .....	<code>nag_rngs_orthog_matrix</code>	(g05qac)
random permutation of an integer vector .....	<code>nag_rngs_permute</code>	(g05nac)
random sample from an integer vector .....	<code>nag_rngs_sample</code>	(g05nbc)
random table .....	<code>nag_rngs_2_way_table</code>	(g05qdc)

Generation of time series,

asymmetric GARCH Type II .....	<code>nag_generate_agarchII</code>	(g05hlc)
asymmetric GJR GARCH .....	<code>nag_generate_garchGJR</code>	(g05hmc)
symmetric GARCH or asymmetric GARCH Type I .....	<code>nag_generate_agarchI</code>	(g05hkc)

univariate ARMA model,

Normal errors .....	<code>nag_rngs_arma_time_series</code>	(g05pac)
---------------------	--	----------

vector ARMA model,

Normal errors .....	<code>nag_rngs_varma_time_series</code>	(g05pcc)
---------------------	---	----------

Pseudo-random numbers,

array of variates from multivariate distributions,

multinomial distribution .....	<code>nag_rngs_gen_multinomial</code>	(g05mrc)
Normal distribution .....	<code>nag_rngs_matrix_multi_students_t</code>	(g05lxc)
Student's <i>t</i> distribution .....	<code>nag_rgsn_matrix_multi_normal</code>	(g05lyc)

Copulas

Gaussian Copula .....	<code>nag_rngs_copula_normal</code>	(g05rac)
Student's <i>t</i> Copula .....	<code>nag_rngs_copula_students_t</code>	(g05rbc)

initialize generator,

nonrepeatable sequence .....	<code>nag_rngs_init_nonrepeatable</code>	(g05kcc)
repeatable sequence .....	<code>nag_rngs_init_repeatable</code>	(g05kbc)

single variate from multivariate distributions,

Normal distribution .....	<code>nag_rngs_multi_normal</code>	(g05lzc)
---------------------------	------------------------------------	----------

single variate, from a univariate distribution

logical value <b>Nag_True</b> or <b>Nag_False</b> .....	<code>nag_rngs_logical</code>	(g05kec)
real number from the continuous uniform distribution .....	<code>nag_rngs_basic</code>	(g05kac)

vector of variates from discrete univariate distributions,

binomial distribution .....	<code>nag_rngs_binomial</code>	(g05mjc)
geometric distribution .....	<code>nag_rngs_geom</code>	(g05mbc)
hypergeometric distribution .....	<code>nag_rngs_hypergeometric</code>	(g05mlc)
logarithmic distribution .....	<code>nag_rngs_logarithmic</code>	(g05mdc)
negative binomial distribution .....	<code>nag_rngs_neg_bin</code>	(g05mcc)
Poisson distribution .....	<code>nag_rngs_poisson</code>	(g05mkc)
uniform distribution .....	<code>nag_rngs_discrete_uniform</code>	(g05mac)
user-supplied distribution .....	<code>nag_rngs_gen_discrete</code>	(g05mzc)

variate array from discrete distributions with array of parameters,

Poisson distribution with varying mean .....	<code>nag_rngs_compound_poisson</code>	(g05mec)
--	--	----------

vectors of variates from continuous univariate distributions,

beta distribution .....	<code>nag_rngs_beta</code>	(g05lec)
Cauchy distribution .....	<code>nag_rngs_cauchy</code>	(g05llc)
chi-square distribution .....	<code>nag_rngs_chi_sq</code>	(g05lcc)
exponential mix distribution .....	<code>nag_rngs_exp_mix</code>	(g05lqc)
<i>F</i> -distribution .....	<code>nag_rngs_f</code>	(g05ldc)
gamma distribution .....	<code>nag_rngs_gamma</code>	(g05lfc)
logistic distribution .....	<code>nag_rngs_logistic</code>	(g05lnc)

lognormal distribution .....	nag_rngs_lognormal	(g051kc)
negative exponential distribution .....	nag_rngs_exp	(g051jc)
Normal distribution .....	nag_rngs_normal	(g051ac)
Student's <i>t</i> -distribution .....	nag_rngs_students_t	(g051bc)
triangular distribution .....	nag_rngs_triangular	(g051hc)
uniform distribution .....	nag_rngs_uniform	(g051gc)
von Mises distribution .....	nag_rngs_von_mises	(g051pc)
Weibull distribution .....	nag_rngs_weibull	(g051mc)
Quasi-random numbers,		
Normal distribution .....	nag_quasi_random_normal	(g05ybc)
uniform distribution .....	nag_quasi_random_uniform	(g05yac)
Superseded functions,		
pseudo-random integer(s),		
reference vector,		
binomial distribution .....	nag_ref_vec_binomial	(g05edc)
cumulative distribution .....	nag_ref_vec_discrete_pdf_cdf	(g05exc)
Poisson distribution .....	nag_ref_vec_poisson	(g05ecc)
Poisson distribution or binomial distribution .....	nag_return_discrete	(g05eyc)
probability distribution .....	nag_ref_vec_discrete_pdf_cdf	(g05exc)
uniform distribution .....	nag_random_discrete_uniform	(g05dyc)
pseudo-random real number(s),		
beta distribution .....	nag_random_beta	(g05fec)
gamma distribution .....	nag_random_gamma	(g05ffc)
multivariate Normal vector,		
reference vector .....	nag_ref_vec_multi_normal	(g05eac)
return multivariate Normal vector .....	nag_return_multi_normal	(g05ezc)
uniform distribution over (0,1) .....	nag_random_continuous_uniform	(g05cac)
uniform distribution over ( <i>a</i> , <i>b</i> ) .....	nag_random_continuous_uniform_ab	(g05dac)
pseudo-random real numbers, from continuous distributions:		
single random number:		
exponential distribution .....	nag_random_exp	(g05dbc)
Normal distribution .....	nag_random_normal	(g05ddc)
sampling and permutation,		
pseudo-random permutation .....	nag_ran_permut_vec	(g05ehc)
pseudo-random sample .....	nag_ran_sample_vec	(g05ejc)
state functions,		
initialize generator,		
nonrepeatable sequence .....	nag_random_init_nonrepeatable	(g05ccc)
repeatable sequence .....	nag_random_init_repeatable	(g05cbc)
restore generator state .....	nag_restore_random_state	(g05cgc)
save generator state .....	nag_save_random_state	(g05cfc)
Time series,		
ARMA .....	nag_arma_time_series	(g05hac)

## 5 Functions Withdrawn or Scheduled for Withdrawal

None.

## 6 References

Boye E (Unpublished manuscript) Copulas for Finance: A reading guide and some applications Financial Econometrics Research Centre, City University Business School, London

Knuth D E (1981) *The Art of Computer Programming (Volume 2)* (2nd Edition) Addison–Wesley

Maclaren N M (1989) The generation of multiple independent sequences of pseudorandom numbers *Appl. Statist.* **38** 351–359

Morgan B J T (1984) *Elements of Simulation* Chapman and Hall

Nelsen R B (1998) *An Introduction to Copulas. Lecture Notes in Statistics 139* Springer

Ripley B D (1987) *Stochastic Simulation* Wiley

Sklar A (1973) Random Variables: Joint Distribution Functions and Copulas *Kybernetika* **9** 499–460

---